IWASAWA'S CONTROL THEOREM ON $\mathbb{Z}_p$-EXTENSIONS

Notations

* F number field
* F_∞/F $\mathbb{Z}_p$-extension if $\mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p$.

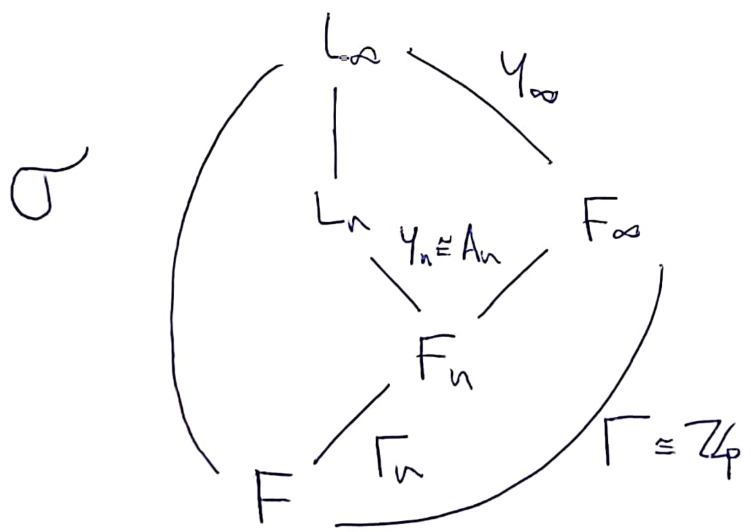
  ↳ For all n, $\exists!$ $\overbrace{F_n}^{\text{subextension}}$ st $[F_n : F] = p^n$.

  These extension, together with F_∞, are the fields between F and F_∞.

Ramification

(a) l prime of F not above p. Then, F_∞/F is unramified at l.

(b) At least one prime ramifies in F_∞/F and $\exists\, n \geq 0$ st every prime which ramifies in F_∞/F_n is totally ramified.



* A_n p-Sylow of $Cl(F_n)$
* $\mathrm{Gal}(L_n/F_n) \cong A_n$, L_n/F_n abelian unramified. (L_n Galois over F)
* $L_\infty = \bigcup_{n \geq 0} L_n$
* γ_0 top. generator of Γ.

$\Lambda(\Gamma) = \varprojlim \mathbb{Z}_p[\mathrm{Gal}(F_n/F)]$, $\Lambda(\Gamma) \cong \Lambda$.

Y_∞ is a Λ-module: for $\gamma \in \Gamma$, lift it to $\tilde{\gamma} \in \sigma = \mathrm{Gal}(L_\infty/F)$ and define $\gamma \cdot y = \tilde{\gamma} y \tilde{\gamma}^{-1}$.

<u>Theorem</u> (to be proved) $\exists\, \lambda, \mu \geq 0$ integers, $\nu \in \mathbb{Z}_p$ and n_0 st $|A_n| = p^{\mu p^n + \lambda n + \nu}$.

<u>Idea</u> Show that Y_∞ is a finitely generated Λ-module and apply the structure theorem.

↳ Show it first under the assumption that all primes which ramify in F_∞/F are totally ramified.

<u>Observations</u>

- $Y_\infty = \varprojlim Y_n$
- $\{p_1, \dots, p_s\}$ primes ramifying in F_∞/F, and $\tilde{p}_i$ primes of L_∞ above p_i. I_i inertia group.
  Then, $\sigma = I_i Y_\infty = Y_\infty I_i$ $\forall\, i = 1, \dots, s$.
  $\sigma_i \in I_i$ mapping to γ ($I_i \cong \mathbb{Z}_p$). $\exists\, a_i \in Y_\infty$ w/ $\sigma_i = a_i \sigma_1$.

<u>Lemma 1</u> G' closure of commutator of σ. Then, $G' = Y_\infty^{\gamma - 1} = T Y_\infty$

<u>Lemma 2</u> $Z_0 = \langle Y_\infty^{\gamma_0 - 1}, a_2, \dots, a_s \rangle \subset Y_\infty$, $Z_n = \nu_n Z_0$, w/ $\nu_n = 1 + \gamma_0 + \dots + \gamma_0^{p^n - 1} = \frac{(1+T)^{p^n} - 1}{T}$.
Then, $Y_n \cong Y_\infty / Z_n$ for all $n \geq 0$.

<u>Proposition</u> $Y_\infty = \mathrm{Gal}(L_\infty/F_\infty)$ is f.g. as a Λ-module.
↳ Key point: lemma 2 + Nakayama