TCC COURSE ON IWASAWA THEORY
ASSIGNEMENT 2
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This is the second of 3 problem sheets for this course, covering material from lectures 3 and 4. Questions are assessed out of a maximum of 100 (although the sum is indeed 120). Students taking this course for credit should submit their solutions to me (by email) by **noon on Monday 22nd November**.
This list has three distinct parts, so you can choose which kind of problems are more related with your preferences. The first part (35 points) asks you to go through the results covered during the lectures and fill out some details we skipped. The second part (65 points) consists on applications of the Iwasawa's control theorem. The last one (20 points) discusses some related results in the specific case of CM fields.
Recall that you only need a minimum of 40 points in each problem sheet.

1 Review of the lectures

These problems ask you to prove some results we used during the lectures, but whose proof we deliberately skipped.

Problem 1 (15 points)**.** Under the assumption that all primes which are ramified in F_∞/F are totally ramified, show the following result.
Let Z_0 be the $\mathbb{Z}_p$-submodule of Y_∞ generated by $\{a_i \mid 2 \leq i \leq s\}$ and by $Y_\infty^{\gamma_0 - 1} = TY_\infty$. Let $Z_n = \nu_n Z_0$, where

$$\nu_n = 1 + \gamma_0 + \ldots + \gamma_0^{p^n-1} = \frac{(1+T)^{p^n} - 1}{T}.$$

Then, $Y_n \cong Y_\infty / Z_n$ for all $n \geq 0$.
Explain with your own words the role of this result in the proof of Iwasawa's control theorem.
Indication. Consider first the case $n = 0$, and show that $\mathrm{Gal}(L_\infty/L_0)$ is generated by G' and all the inertia groups I_i, $1 \leq i \leq s$ (with the notations used during the lectures).

Problem 2 (10 points)**.** Show the following version of Nakayama's lemma: *Let X be a compact Λ-module. Then, X is finitely generated over Λ if and only if $X/(p,T)X$ is finite. Further, if $x_1, \ldots, x_n$ generate $X/(p,T)X$ over $\mathbb{Z}$, they also generate X as a Λ-module.*
Explain with your own words the role of this result in the proof of Iwasawa's control theorem.
Indication. Show that for any small neighbourhood U of 0 in X, there exists a positive integer n such that $(p,T)^n X \subset U$.

Problem 3 (10 points)**.** Let $\mathcal{F} = \mathbb{Q}(\mu_p)$ and consider its cyclotomic $\mathbb{Z}_p$-extension, $\mathcal{F}_\infty/\mathcal{F}$. Write $\mathcal{G}$ for the Galois group $\mathrm{Gal}(\mathcal{F}_\infty/\mathbb{Q})$.

(a) Justify that $\mathcal{G} \cong \Delta \times \Gamma$, where $\Delta \cong \mathbb{Z}/(p-1)\mathbb{Z}$.

(b) Show that $\Lambda[\Delta] = \mathbb{Z}_p[[T]][\Delta]$ is isomorphic to $\Lambda(\mathcal{G}) = \lim_{\leftarrow} \mathbb{Z}_p[\mathrm{Gal}(\mathcal{F}_n/\mathbb{Q})]$.

(c) Let $f \in \mathbb{Z}_p[[T]][\Delta]$, and let $\sigma \in \mathcal{G}$. Show that the action of $\mathcal{G}$ on $\mathbb{Z}_p[[T]][\Delta]$ induced by the previous isomorhpism is given by

$$(\sigma f)(T) = f((1+T)^{\chi(\sigma)} - 1).$$

2 Applications of Iwasawa's control theorem

Problem 4 (10 points)**.** (a) Show that in a cyclotomic $\mathbb{Z}_p$-extension no prime splits completely.

(b) Suppose that $\mathrm{Gal}(K/F) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Let $\ell \neq p$ be a prime. Show that the decomposition group of a prime $\tilde{\ell}$ above ℓ is either trivial or isomorphic to $\mathbb{Z}_p$. *Hint:* look at Frobenius.

(c) Conclude that there is a subextension $K_1 \subset K$ such that $\mathrm{Gal}(K_1/F) \simeq \mathbb{Z}_p$ and such that some prime above ℓ splits completely in K_1/F.

Problem 5 (5 points). (a) Show that if $\sqrt{2} \notin F$, then $F(\sqrt{2})/F$ is the first step of the cyclotomic $\mathbb{Z}_2$-extension of F.

(b) Show that $\mathbb{Q}(\sqrt{-6}, \sqrt{2})/\mathbb{Q}(\sqrt{-6})$ is unramified. Hence, it is possible that F_1/F_0 is unramified in a $\mathbb{Z}_p$-extension.

Problem 6 (15 points). Let F be a number field and F_∞/F a $\mathbb{Z}_p$-extension of F. For $n \geq 0$, let F_n/F denote the unique subextension of F_∞/F that has degree p^n over F, and set $\Gamma_n = \mathrm{Gal}(F_\infty/F_n)$. Let A_n denote the p-Sylow subgroup of the ideal class group of F_n. Let L_∞ denote the maximal abelian unramified pro-p extension of F_∞ and let $Y_\infty := \mathrm{Gal}(L_\infty/F_\infty)$ denote its Galois group over F_∞.

(a) Prove that $\mu(Y_\infty) = 0$ if and only if the sequence $(\dim_{\mathbb{F}_p} A_n/pA_n)_n$ is bounded as n tends to infinity.

(b) Calculate $\mu(Y_\infty)$ and $\lambda(Y_\infty)$ for $F = \mathbb{Q}$ (and for the obvious choice of F_∞).

(c) Suppose there is only one prime of F that ramifies in F_∞/F and that this prime is totally ramified. Prove that $(Y_\infty)_{\Gamma_n} \simeq A_n$ and that $A_0 = 0$ if and only if $A_n = 0$ for all $n \geq 0$.

(d) Find a number field F and a prime p such that the ν-invariant for the cyclotomic $\mathbb{Z}_p$-tower is non-zero.

Problem 7 (15 points). Suppose $\Gamma \simeq \mathbb{Z}_p$ and for each non-negative integer n, set $\Gamma_n := \Gamma^{p^n}$. Let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ and let M be a finitely generated torsion Λ-module with characteristic polynomial f. Prove that the following three conditions are equivalent:

(a) M^{Γ_n} is finite.

(b) M_{Γ_n} is finite.

(c) $f(\zeta - 1)$ is non-zero for any $\zeta \in \mu_{p^n}$.

When this is the case, prove that

$$\frac{M^{\Gamma_n}}{M_{\Gamma_n}} = p^{-\mu(M)p^n} \prod_{\zeta \in \mu_{p^n}} |f(\zeta - 1)|_p.$$

Problem 8 (20 points). Let F be an imaginary quadratic field and let p be an odd prime. Let $h_F^{(p)}$ denote the order of the p-Sylow subgroup of the ideal class group of F. Let F_∞/F denote the cyclotomic $\mathbb{Z}_p$-extension of K and Y_∞ the Galois group of the maximal unramified pro-p extension of F_∞ over F_∞. Let $f(T)$ denote any generator of the characteristic ideal of Y_∞.

(a) Suppose that the prime p is either inert or ramified in $F/\mathbb{Q}$.

  (i) Prove that $f(0) \neq 0$.

  (ii) Prove that $\mathrm{ord}_p\, h_F^{(p)} = \mathrm{ord}_p\, f(0)$.

(b) Suppose that the prime p is split in $F/\mathbb{Q}$.

  (i) Prove that $f(T)$ has a simple zero at $T = 0$.

  (ii) Suppose that $F = \mathbb{Q}(i)$. Calculate $f'(0)$.

  (iii) More generally, calculate the ratio $f'(0)/h_F^{(p)}$.

3 CM fields

Recall that a number field K is a CM field if it is a quadratic extension K/F where F is totally real and K is totally imaginary (every embedding of F into $\mathbb{C}$ lies entirely within $\mathbb{R}$, but there is no embedding of K into $\mathbb{R}$). If each F_n is a CM field, then F_∞^+/F^+ is a $\mathbb{Z}_p$-extension. If p is odd we may decompose A_n as $A_n = A_n^+ \oplus A_n^-$. Further, $Y_n = Y_n^+ \oplus Y_n^-$ and hence $Y = Y^+ \oplus Y^-$. Proceeding in the same way as in the lectures, $A_n^\pm \simeq Y_n^\pm \simeq Y_\infty^\pm/\nu_{n,e}Y^\pm$. If $p^{e_n^\pm}$ is the exact power of p dividing $h_n^\pm$, then $e_n = e_n^+ + e_n^-$ and

$$e_n^\pm = \lambda^\pm n + \mu^\pm p^n + \nu^\pm, \qquad \text{for } n \geq n_0^\pm.$$

Those items introduced with (-) and not with a letter can be skipped, and you can use the result for the next parts of the problems. They are supposed to require more advanced knowledge for its resolution, but you are invited to try to solve them (or look at the course references, in this case Washington's book).

Problem 9 (15 points)**.** (-) Let L be a CM field with $\zeta_p \in L$, and let A be the p-Sylow of the class group of A. Show that

$$\text{p-rank } A^+ \leq 1 + \text{p-rank } A^-.$$

(a) Let p be a prime. Suppose K is a CM field with ζ_p in K, and let K_∞/K be the cyclotomic $\mathbb{Z}_p$-extension. Then, $\mu = 0$ if and only if $\mu^- = 0$.

(b) Suppose K_∞/K is a $\mathbb{Z}_p$-extension with $\mu = 0$. Then,

$$Y_\infty \simeq \varprojlim A_n \simeq \mathbb{Z}_p^\lambda \oplus \text{ finite p-group}$$

as $\mathbb{Z}_p$-modules. Does the same result hold for Y_∞^-?

(c) Show that the previous result is not necessarily true when $p = 2$. *Hint: show that the ideal* $(2, \sqrt{-6})$ *is not principal in* $\mathbb{Q}(\sqrt{-6})$*, but is principal in* $\mathbb{Q}(\sqrt{-6}, \sqrt{2})$.

(-) Let p be odd, and suppose K is a CM field, with K_∞/K its cyclotomic $\mathbb{Z}_p$-extension. Show that the map $A_n^- \longrightarrow A_{n+1}^-$ is injective.

(d) Let p be odd, let K be a CM field and let K_∞/K be the cyclotomic $\mathbb{Z}_p$-extension. Show that $Y_\infty^- = \lim_\rightarrow A_n^-$ contains no finite Λ-modules, and that there is an injection with finite cokernel

$$Y_\infty^- \hookrightarrow \bigoplus_i \Lambda/(p^{k_i}) \oplus \bigoplus_j \Lambda/(g_j(T)).$$

(e) Let p be odd, and let K be a CM field with cyclotomic $\mathbb{Z}_p$-extension K_∞/K. If $\mu^- = 0$, show that $Y_\infty^- \simeq \mathbb{Z}_p^{\lambda^-}$.

Class field theory implies that the number of independent $\mathbb{Z}_p$-extensions of a number field F is $r_2 + 1 + \delta$, where r_2 is the number of pairs of complex conjugate embeddings of R and $\delta \geq 0$ is called the Leopoldt's defect. You can freely use this result without proof (see Section 13.4 in Wasington's book for a quite elementary proof). Leopoldt's conjecture asserts that $\delta = 0$. It has been proved when $F/\mathbb{Q}$ is abelian.

Problem 10 (5 points)**.** Suppose L/K is a finite extension of number fields.

(a) If Leopoldt's conjecture is valid for L and p, prove that it also holds true for K and p.

(b) Suppose L is a CM extension and $K := L^+$ is its maximal totally real subfield. If the Leopoldt conjecture holds for K and p, prove that it is also true for L and p.

(c) Prove that Leopoldt's conjecture for abelian extensions of $\mathbb{Q}$ follows from Leopoldt's conjecture for totally real abelian extensions.